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ON SPECIALIZATIONS OF CURVES. I

BY

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ABSTRACT. The following is proved: Given a family of projective reduced curves $X \to T$ (T irreducible), if X_t (the general curve) is integral and X_0 is a special curve (having irreducible components $X_1, \ldots, X_r$), then $\Sigma_{i=1}^r g_i(X_i) \leq g(X_t)$, where $g(Z)$ = geometric genus of Z. Conversely, if A is a reduced plane projective curve, of degree n with irreducible components $X_1, \ldots, X_r$, and g satisfies $\Sigma_{i=1}^r g_i(X_i) \leq g \leq \frac{1}{2}(n-1)(n-2)$, then a family of plane curves $X \to T$ (with T integral) exists, where for some $t_0 \in T$, $X_{t_0} = Z$ and for t generic, X_t is integral and has only nodes as singularities. Results of this type appear in an old paper by G. Albanese, but the exposition is rather obscure.

In 1928 Giacomo Albanese published a beautiful paper (cf. [**1**]), where he presents some very interesting results on families of plane curves. In it, he gives necessary and sufficient conditions in order that a plane curve B (not necessarily reduced, i.e., an effective divisor in $\mathbf{P}^2$, $B = k_1B_1 + \cdots + k_sB_s$, where $B_1, \ldots, B_s$ are the irreducible components of B_{red}) could be a specialization of an integral curve $A \subset \mathbf{P}^2$ (conditions in terms of the geometric genera of A and B_i, $i = 1, \ldots, s$, as well as the coefficients k_i). Unfortunately, even for the standards of that period, the paper is rather obscurely written. Perhaps I should quote Zariski's commment in [**12**, p. 216] "... but the conclusions... cannot be regarded as final in view of the extreme delicacy of the arguments involved". In spite of the many years that have gone by, apparently little or nothing appears in the modern literature in this direction. Of course, one has some deep results on degeneration of smooth curves into curves with certain types of nodes (stable curves, cf. [**2**]), but this theory is, probably, of a quite different nature.

In this paper, I make a modest contribution to the theory, namely I give a modern (and, I hope, rigorous) presentation of some of the results claimed by Albanese. These have to do with families of *reduced* curves, i.e., in the notation just introduced, in the case where $k_i = 1$ for all i. The precise statements of the main results are in §1, namely Theorems (1.2) (on how the genus must behave, given a family) and (1.4) (on existence of a family or specialization, when certain numerical conditions are satisfied).

The proof of Theorem (1.2) given here is entirely different from Albanese's. The proof of (1.4) is very similar to his, actually it is his proof, with the verification of a number of details added. Curiously enough, to verify some details of this 1928 proof,

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one has to use some results from the very recent paper [**10**] by O. Zariski. Albanese's proof, which is a masterpiece of geometric imagination, makes a clever use of Severi's variety of curves with nodes [**8**, Anhang F] (fortunately, of parts of Severi's theory that can be verified, cf. [7]). One is tempted to prove the existence Theorem (1.4) by means of deformation theory, but that seems difficult because of obstructions. For instance, if C is a reduced curve in $\mathbf{P}^2$, J its jacobian ideal, then the fundamental sequence relating local deformations of the singularities and first-order deformations of C (inside $\mathbf{P}^2$) is

$$H^0(C, N_C) \xrightarrow{\alpha} H^0(C, N_C \otimes \mathcal{O}_S) \xrightarrow{\beta} H^1(C, J \otimes N_C) \to 0$$

where N_C is the normal bundle; note $H^0(C, N_C \otimes \mathcal{O}_S) \approx \oplus_{p \in S} T^1_{C,p}$, S being the singular set of C. But $H^1(C, J \otimes N_C)$ could be nonzero, so one should carefully study β, which probably is not easy, if the singularities of C are complicated.

In §0 some preliminaries on notation and auxiliary concepts are presented, in §1 the main statements are given. The proof of Theorem (1.2) is in §2. The proof of (1.3) is in §§4 and 5 (in §4 an auxiliary theorem, namely, Theorem (1.4), interesting in itself, is presented). In §3, some necessary concepts are reviewed. In particular, for the reader's convenience, a summary of facts from "Anhang F" that we need is given.

As pointed out before, more general results than those discussed here are claimed in [**1**]. There, the curve B is not assumed to be reduced. Also, some geometric properties of the singularities of B, which are "limits" of the nodes of the general A, are claimed.

I intend to discuss some of these finer items in a future article.

0. Preliminaries.

(0.1) The basic terminology is taken from [**4**], with some minor changes to be explained next. However, we need only the more elementary aspects of the theory of schemes, since we shall always work with schemes over a fixed algebraically closed field k.

(0.2) More precisely, we work in the category $\mathcal{AS}(k)$ of algebraic schemes of finite type over a field k, *algebraically closed* and of *characteristic zero*. An *algebraic variety* is a *reduced* scheme in $\mathcal{AS}(k)$ (not necessarily irreducible). If Z is an algebraic variety, Sing(Z) is the singular locus of Z. We write $\mathbf{P}^m$ to denote the m-dimensional projective space over k.

(0.3) If T is an integral scheme (in $\mathcal{AS}(k)$), the term "a (or the) general (or generic) point of T" will mean a closed point, taken from a suitable open, nonempty set in T, which will be either clear from the context or explicitly specified. This is done trying to make the presentation more geometric. We believe that the interested reader can easily translate any argument where this convention is used into one where the expression "generic point" is used in the strict sense of the theory of schemes.

(0.4) A *curve* is an algebraic variety of pure dimension one. A *plane curve* is a closed subscheme of $\mathbf{P}^2$ which is a curve. A *nodal curve* is a reduced plane curve, such that all its singular points are ordinary double points or nodes. The terms "geometric genus" and "arithmetic genus" of a reduced curve C are used as in [**4**], and denoted by $g(C)$ and $p_a(C)$, respectively (if C has irreducible components $C_1, \ldots, C_r$, then $g(C)$ means $\sum_{i=1}^r g(C_i)$).

If $p \in C$ (a reduced curve), its "δ invariant" $\delta(p, C)$ is the length of the $\mathcal{O}_{C,p}$-module $\overline{\mathcal{O}}_{C,p}/\mathcal{O}_{C,p}$, where a bar indicates normalization.

(0.5) Recall the "genus formula". Given a connected, reduced curve Z, then

$$p_a(Z) = g(Z) + \sum_{p \in Z} \delta(p, Z) - r + 1 \tag{0.5.1}$$

where r is the number of irreducible components of Z.

(0.6) A *family* of *curves* is a flat morphism f: $X \to T$ (in $\mathcal{AS}(k)$), such that for each closed point $t \in T$, $f^{-1}(t)$ is a curve. The family is (a) *proper* if f is a proper morphism, and (b) of reduced curves if $f^{-1}(t)$ is reduced for all $t \in T$. A family of plane curves (parametrized by T) is an effective relative Cartier divisor $\mathfrak{D} \subset \mathbf{P}^2 \times T$ (then if f: $\mathfrak{D} \to T$ is induced by the second projection, this is a family in the sense just introduced).

Sometimes we may talk about the "generic" genus of a family of curves. Precisely, we have

(0.7) LEMMA. *Let f: $X \to T$ be a proper family of reduced curves, parametrized by an irreducible scheme T, such that the general curve is irreducible. Then, there is a nonnegative integer p and an open $U \neq \varnothing$ of T, such that for each $t \in T$, X_t is an integral curve of geometric genus p.*

PROOF. By pulling-back over T_{red}, and then to the open of nonsingular points of T_{red}, we may assume T integral and smooth. Then, since f is flat and the generic fiber is integral, it follows that X is integral. Take the normalization q: $Z \to X$ of X, let $\pi = f \cdot q$. Then, $\dim Z = \dim T + 1$, $\operatorname{Sing}(Z) = S$ has codimension $\leqslant 2$, and $\pi(S)$ has codimension $\leqslant 1$. Apply Corollary 10.7 of [**4**, p. 272] to the induced morphism: $\pi^{-1}(T - S') \to T - S'$ with $S' = \pi(S)$. It follows for an open V of T, if $t \in V$ then $h^{-1}(t) = \pi^{-1}(t) = Z_t$ is smooth and for a suitable open $U \subset V$, Z_t is the normalization of X_t (for the latter, apply again Corollary 10.7 of [**4**] to X-$\operatorname{Sing}(X) \to T$). Thus, for $t \in U$, $g(X_t) = p_a(Z_t)$. But since we may assume that $\pi^{-1}(U) \to U$ is flat (by generic flatness), $p_a(Z_t)$ is constant for $t \in U$.

(0.8) DEFINITION. Given a proper family of reduced curves f: $X \to T$, with T irreducible, the *generic genus* of the family is $g(X_t)$, for $t \in U$, where U is an open as that of Lemma (0.7).

1. Statement of the main results.

(1.1) Next, we state the main results to be proved in this article.

Throughout, we work with an algebraically closed field k of characteristic zero.

(1.2) THEOREM. *Let f: $X \to T$ be a proper family of reduced curves, where the general curve is irreducible, of geometric genus g (cf. (0.7)), and T is an irreducible algebraic k-scheme. Let $t_0 \in T$ be a closed point, $f^{-1}(t_0) = B = \bigcup_{i=1}^r B_i$ (where $B_1, \ldots, B_r$ are the irreducible components of B), g_i the geometric genus of B_i, $i = 1, \ldots, r$.*

Then,

$$(1.2.1) \qquad g \geqslant \sum_{i=1}^{r} g_i.$$

Note that f is not assumed to be a family of plane curves.

Theorem (1.2) has the following converse.

(1.3) THEOREM. *Let B be a projective, plane, reduced curve of degree n, having irreducible components $B_1, \ldots, B_r$ (the curve B can have arbitrary singularities). Let g_i be the geometric genus of B_i, and g a given integer, satisfying*

$$(1.3.1) \qquad \sum_{i=1}^{r} g_i \leqslant g \leqslant \frac{(n-1)(n-2)}{2}.$$

Then, there is a family of plane curves

$$\begin{array}{ccc} X & \rightarrow & \mathbf{P}^2 \times T \\ & {}^{f}\searrow \quad \swarrow & \\ & T & \end{array}$$

with T integral, and a closed point $t_0 \in T$, such that (a) $f^{-1}(t_0) = B$ *and* (b) *the general curve is integral, of genus g* (*cf.* (0.4)).

Theorem (1.2) will be a consequence of the following result.

(1.4) THEOREM. *Let B be an integral plane curve of degree n and geometric genus p, having arbitrary singularities. Then, there is a family of plane curves $X \xrightarrow{f} T$, T integral, and $t_0 \in T$ such that $f^{-1}(t_0) = B$ and the general member is an irreducible nodal curve of geometric genus p.*

2. Proof of Theorem (1.2).

(2.1) Here we present a proof of Theorem (1.2). A proof of a similar result, but where the specializations are in the sense of the theory of Chow coordinates and not of flat families, can be seen in [**5**].

(2.2) PROOF OF (1.2). By pulling back over the normalization of a suitable integral curve $T_1 \subset T$, we may assume that T is a smooth, integral curve. Then it easily follows that X must be an integral surface. Take the normalization $\pi: Z \to X$ of X. Now, Z is a normal surface, hence Cohen-Macaulay. Since the composite morphism $p = f \circ \pi: Z \to T$ is equidimensional and T is smooth, it follows that p is flat.

We claim that p defines a family of reduced curves, where, for all $t \in T$, $p^{-1}(t)$ is birationally equivalent to $f^{-1}(t)$ and, for t generic, $f^{-1}(t)$ is smooth. To begin with, since T is a smooth curve, the fibers $f^{-1}(t)$ will be Cartier divisors (defined at $\mathcal{O}_{Z,z}$ by $p^*(t) = 0$, t a parameter of $\mathcal{O}_{T,f(z)}$). Since Z is Cohen-Macaulay, each divisor $Z_t = p^{-1}(t)$ will be Cohen-Macaulay. Similarly, $X_t = f^{-1}(t)$ is a divisor of X. Consider X_t, which is reduced by assumption. In each component of X_t, we may find points x such that X_t is smooth at x. Since X_t is a divisor of X, it follows that X is smooth at x. From this, we may easily find an open U in X such that $U \cap X_t$ is dense in S_t and that the normalization map π induces an isomorphism $\pi^{-1}(U) \xrightarrow{\sim} U$. It follows that $\pi^{-1}(U) \cap Z_t \approx U \cap X_t$, hence Z_t is "generically reduced" and, being Cohen-Macaulay, it is everywhere reduced. The isomorphism above also shows the claimed birationality statement. The assertion "Z_t is smooth for t generic in T" follows from "generic smoothness" (see [**4**, Corollary 10.7, p. 272]), applied to the morphism $Z - p^{-1}(S) \to T - S$, where $S = p(\mathrm{Sing}(Z))$ (note $\mathrm{Sing}(Z)$ is a finite set, since Z is normal).

Now, take t generic in T, and let $Z_0 = Z_{t_0}$ (t_0 is the given point in (1.2)). By flatness, $p_a(Z_t) = p_a(Z_0)$. But Z_t is just the normalization of X_t, hence $p_a(Z_t) = g$ $=$ geometric genus of X_t. On the other hand, Z_0 is connected by the "connectedness principle" (cf. [**4**, p. 281]), hence $p_a(Z_0) = g(Z_0) + \Sigma_{z\in Z_0}\delta(z, Z_0) - (r-1)$, and $g(Z_0) = g(X_0)$ by birationality. But we also have $\Sigma_{z\in Z_0}\delta(z, Z_0) \geq (r-1)$. In fact, this is an easy consequence of the connectedness of Z_0 (e.g., see [**5**, p. 180]). Putting these facts together, we get $g \geq g(X_0) = \Sigma_{i=1}^{r} g_i$, as desired.

3. Review of some useful results. Here we review some known facts that are necessary to prove Theorems (1.3) and (1.4). In (3.1) and (3.2) we review facts about "Severi's variety of nodal curves." For details see [**8**, Anhang F] or (in a modern presentation) [**7**].

(3.1) (a) All plane curves of degree n are parametrized by a $\mathbf{P}^N$, $N = \frac{1}{2}n(n+3)$. More precisely, this is the connected component of the Hilbert scheme of $\mathbf{P}^2$ corresponding to effective divisors of degree n. There is a universal curve $D \subset \mathbf{P}^N \times \mathbf{P}^2$, having equation

$$\Sigma a_{ijl} x_0^i x_1^j x_2^l = 0, \qquad i + j + l = n. \tag{3.1.1}$$

Given any family $X \to T$ of plane curves of degree n, there is a unique morphism $T \to \mathbf{P}^N$, such that $X \to T$ is isomorphic to the pull-back of the universal family.

(b) Points of $\mathbf{P}^N$ corresponding to curves C with exactly δ nodes (i.e., $\mathrm{Sing}(C) = \{P_1, \ldots, P_\delta\}$, and each P_i is a node) form a locally closed subvariety (generally reducible) $U_{n,\delta}$, its closure will be denoted by $V_{n,\delta}^{(0)}$. Each irreducible component of $V_{n,\delta}^{(0)}$ has the same dimension, namely $N - \delta = 3n + p - 1$, $p =$ geometric genus of any curve $C \in U_{n,\delta}$. The variety $V_{n,\delta}^{(0)}$ is the parameter space of a canonical family, $\mathfrak{D}' \to V_{n,\delta}^{(0)}$, namely the pull-back of $\mathfrak{D}$ of (3.1.1) and it has the following universal property: Given a family of curves $X \to T$, T reduced, where general fibers have exactly δ nodes, then the canonical morphism $T \to \mathbf{P}^N$ factors through $V_{n,\delta}^{(0)}$ (actually, it has a finer universal property, more complicated to state, which we omit because we do not need it, see [**7**, §§3 and 4]).

(c) The variety $V_{n,\delta}^{(0)}$ contains all curves C with $\delta' \geq \delta$ nodes, and no other singularity, i.e., $V_{n,\delta}^{(0)} \supset V_{n,\delta'}^{(0)} \supset U_{n,\delta'}$, if $\delta' \geq \delta$. If $C \in U_{n,\delta'}$, then $V_{n,\delta}^{(0)}$ has, at C, *smooth* analytic branches only; and the choice of δ among the δ' nodes of C (say, $P_1, \ldots, P_\delta$) determines a unique such branch $\mathfrak{B}$. Namely, there is exactly one branch, whose tangent space (regarded as a linear subvariety of $\mathbf{P}^N$ passing through C) is $\{D$: D is a plane curve of degree n passing through P_i, $i = 1, \ldots, \delta\}$. Of course, this branch $\mathfrak{B}$ determines a unique irreducible component of $V_{n,\delta}^{(0)}$ containing C. In particular, if C has exactly δ nodes, $V_{n,\delta}^{(0)}$ is smooth at C.

(d) Intuitively, and working over $\mathbf{C}$ (where $\mathfrak{B}$ can be regarded as a "germ"), the general curve A of $\mathfrak{B}$ has exactly δ nodes, which approach $P_1,\dots,P_\delta$ as A approaches C (see (3.2)).

(e) Given the "assignment" $P_1,\dots,P_\delta$ on C (cf. (c)), the general curve of the resulting branch (or component of $V_{n,\delta}^{(0)}$, if one prefers) will be irreducible if and only if $C - \{P_1,\dots,P_\delta\}$ is connnected.

(3.2) Algebraically, assertion (d) about "approaching singularities" can be made precise as follows. Let E be an irreducible component of $V_{n,\delta}^{(0)}$, $C \in \Sigma$ and $Q_1,\dots,Q_\delta$ (not necessarily distinct) singular points of C. Sometimes, we would like to say: "by suitably specializing a general curve of E, its δ nodes approach $Q_1,\dots,Q_\delta$" (or "specialize into $Q_1,\dots,Q_\delta$"). This shall mean the following: (i) There is a morphism $\beta\colon T \to \Sigma$, where T is a connected curve, and a point $t_0 \in T$, such that $\beta(t_0) = C$, $\beta(t)$ is a nodal curve with δ nodes, for $t \neq t_0$. (ii) There are δ sections $\varepsilon_i\colon T \to D$, where $D \to T$ is the pull-back of the universal family over $V_{n,\delta}^{(0)}$; moreover, $\{\varepsilon_1(t_0),\dots,\varepsilon_\delta(t_0)\} = \{Q_1,\dots,Q_\delta\}$ and for $t \neq t_0$, $\{\varepsilon_1(t),\dots,\varepsilon_\delta(t)\} = \operatorname{Sing}(D_t)$.

Under these conditions, given $t_1 \in T$, we also say that the node $\varepsilon_i(t_1)$ of D_{t_1} specializes to Q_i. In [**7**, (4.11)], it is shown that always, given $C \in \Sigma$, certain singularities of C will be obtained by suitably specializing a general curve A of E (and its nodes) in the sense just described. Actually, there the case "each Q_i is a node" is treated, but the methods used work in general. The idea is to cover $V_{n,\delta}^{(0)}$ by a variety $W_{n,\delta}^{(0)}$, which is the closure in $\mathbf{P}^{N+2\delta}$ of systems $(C, P_1,\dots,P_\delta)$, with C nodal, P_i a node of C for each i; the pull-back of the universal family $D' \to V_{n,\delta}^{(0)}$ over $W_{n,\delta}^{(0)}$ admits δ canonical sections.

(3.3) We recall some facts from [**10**]. There is a concept of *equivalence* of singularities of locally plane curves, developed in modern terms by Zariski (see [**4**, p. 393 or **11**]). Analytic isomorphism implies equivalence, but the converse is false. An *equisingularity class* (or just a "*class*") is an element of the set of singular points of plane curves, modulo equivalence.

In [**10**], Zariski proves the following:

(a) Let $f\colon X \to T$ be a family of plane, reduced curves (i.e., $X \subset \mathbf{P}^2 \times T$) with T integral. Then, there is an open set $U \subset T$, and equisingular classes $\mathfrak{S}_1,\dots,\mathfrak{S}_r$, such that, for each closed point of U, $X_t = f^{-1}(t)$ is a curve with exactly r singular points which, if suitable ordered, will be of class $\mathfrak{S}_1,\dots,\mathfrak{S}_r$, respectively. Since the geometric genus of X_t depends on the class of singular points, this gives another proof of (0.7).

(b) In particular, this applies to the case where $T \subset \mathbf{P}^N$ (N as in (3.1)), T is integral, $f\colon X \to T$ is the pull-back to T of the universal family in $\mathbf{P}^2 \times \mathbf{P}^N$, given by (3.1.1). In this situation, we have: Let p be the generic geometric genus of the family. Then, $\dim T \leqslant 3n + p - 1$, and the equality holds if and only if the generic member of the family is a nodal curve (and the closure of T is an irreducible component of $V_{n,\delta}^{(0)}$).

(c) In the same paper (p. 224), Zariski shows the following, which will be useful in §5: Consider (notation of (3.1)) $V_{n_1,\delta_1}^{(0)},\dots,V_{n_q,\delta_q}^{(0)}$, for various numbers $n_1,\dots,\delta_q$. Let C_i be a general member of $V_{n_i,\delta_i}^{(0)}$. Then, $C_1 \cup \cdots \cup C_q$ is a curve of degree $n_1 + \cdots + n_q$ with nodes only (i.e., the C_i's meet traversally).

(3.4) We briefly discuss an obvious generalization, to families, of the classical process of taking a "quadratic transform" of the plane.

Consider a family $X \xrightarrow{f} T$, $X \subset \mathbf{P}^2 \times T$, T integral, of plane curves of degree d. Let $t_0 \in T$ (a closed point), $B = X_{t_0} \subset \mathbf{P}^2 \times \{t_0\} \sim \mathbf{P}^2$, and P_0, P_1, P_2 points of $\mathbf{P}^2$. We shall assume that for some neighborhood V of t_0, the morphisms $s_i\colon U \to \mathbf{P}^2 \times U$ given, on closed points, by $s_i(u) = (P_i, u)$, $i = 0, 1, 2$, factor through X (i.e., locally at t_0, s admits the trivial sections at P_0, P_1, P_2, respectively). Then we may define the standard Cremona transformation of this family. Namely, choose coordinates in $\mathbf{P}^2$ so that $P_0 = (1:0:0)$, $P_1 = (0:1:0)$, $P_2 = (0:0:1)$, and use the equations $x_0 = x_1'x_2'$, $x_1 = x_0'x_2'$, $x_2 = x_0'x_1'$. Precisely, we may find an affine open $U = \operatorname{Spec}(A)$ containing t_0, such that, over U the restriction X_A of X ($X_A \subset \mathbf{P}_A^2$) is defined by an equation $F(x) = \sum a_{ijl} x_0^i x_1^j x_2^l = 0$, $i + j + l = n$, $a_{ijl} \in A$. Then the algebraic transform of X_A is given by $F(x_1'x_2', x_0'x_2', x_0'x_1') = 0$. If, for each closed point $t \in U$, X_t has at P_i a point of multiplicity n_i, we may define the proper transform of X_A (also referred to as the standard Cremona transformation of X near t_0) to be given in $\mathbf{P}_A^2$ by $G = 0$, where $G = F/x_0^{n_0}x_1^{n_1}x_2^{n_2}$. Then, $\deg G = 2n - n_0 - n_1 - n_2$.

The basic properties of the classical standard Cremona transform for one curve holds for families under the assumption made above. The proofs are essentially the classical ones (see [**3**, Chapter 7, §4]).

4. Proof of Theorem (1.4).

(4.1) Theorem (1.4) will be an easy consequence of Lemmas (4.2) and (4.3) to be given next.

(4.2) LEMMA. *Statement* (1.4) *is true if we make the additional assumption that* $n > 2p$.

(4.3) LEMMA. *Let B be as in* (1.4). *Choose points P_0, P_1, P_2 not in B, not on a line, and such that each line determined by two of them cuts B at n distinct points. Consider the standard Cremona transformation of the plane with respect to these points, and the proper transform B_1 of B (which, in this case, agrees with the algebraic one). Assume that the conclusion of* (1.4) *holds for B_1. Then, it also holds for B.*

(4.4) PROOF OF (1.4) (ACCEPTING THE LEMMAS). It is by induction on the number $b(B) = \inf\{b/2^b n > 2p\}$ (where n and p are the degree and geometric genus of B, respectively). The case $b(B) = 0$ is just Lemma (4.2). For the inductive step, if $b(B) = m > 0$ and the theorem is valid when $b < m$, take the standard Cremona transformation B_1 of B, which has degree $2n = n_1$, and $2^{m-1}n_1 = 2^m n > p$. So, (1.4) is valid for B_1, and by (4.4) it will be valid for B.

(4.5) PROOF OF LEMMA (4.2). Consider, on $B \subset \mathbf{P}^2$ (of degree n and geometric genus p) the linear series g_n^2 cut out by the lines of $\mathbf{P}^2$, i.e., by sections in $H^0(B, \mathcal{O}_B(1))$. Take the normalization $q\colon \bar{B} \to B$ of B, and look at the linear series $\tilde{g}_n^2$ induced by g_n^2 on $\bar{B}$. If $D \in \tilde{g}_n^2$, then D is nonspecial (because $\deg D = n > 2p$ by assumption), hence $|D|$ has dimension $n - p$; moreover, since $n > 2p$, $|D|$ is very ample (see [**4**, p. 308]). Thus, $|D|$ defines an embedding of $\bar{B}$ into $\mathbf{P}^{n-g}$, i.e., an isomorphism $h\colon \bar{B} \xrightarrow{\sim} C \subset \mathbf{P}^{n-g}$. More canonically, $\tilde{g}$ corresponds to sections of

$H^0(\bar{B}, \mathcal{E})$, where $\mathcal{E} = q^*(\mathcal{O}_B(1))$. Now we have a natural inclusion

$$j\colon H^0(B, \mathcal{O}_B(1)) \xrightarrow{j} H^0\big(B, q_*\mathcal{E}\big) \approx H^0(\bar{B}, \mathcal{E}) \quad \text{(because } q \text{ is finite)},$$

$\mathbf{P}^{n-g}$ is Proj($H^0(\bar{B}, \mathcal{E})$), and the composition $g \circ h^{-1}\colon C \to B$ is induced from the rational map $\rho\colon \mathbf{P}^{n-g} \to P^2$ corresponding to j. Clearly ρ is the projection (on $L = \text{Proj}(\text{Sym } H^0(B, \mathcal{O}_B, (1))^*) \sim \mathbf{P}^2$) from $S_0 = \text{Proj}(\text{Sym } K)$, where $K = \text{Ker}(j^*)$ (j^* is the dual of j). Note: $\dim S_0 = n - p - 3$. Now we let S_0 vary inside $\mathbf{P}^{n-g}$, i.e., in $\text{Grass}(n - g + 1, n - p - 2) = G$. If $\mathcal{U} = \{\mathcal{S} \in G/S \cap L = \varnothing\}$, then for a general element S of $\mathcal{U}$, the projection of C on L from S will be a nodal curve. In this way we obtain a family (parametrized by $\mathcal{U}$), having the required properties. This proves (4.2).

(4.6) Proof of (4.3). Consider B_1, the standard Cremona transform of B. Since P_0, P_1, P_2 are not in B, B_1 has degree $n_1 = 2n$ and $g(B_1) = g(B) = p$, because both are birationally equivalent. Moreover, if we write $p = \frac{1}{2}(n-1)(n-2) - d$, then also $p = \frac{1}{2}(n_1 - 1)(n_1 - 2) - d_1$, with $d_1 = 3n(n-1)/2 + d$, as a simple calculation shows. We are assuming the following: Using the notation of (3.1), there is an irreducible component Σ_1 of $V^{(0)}_{n_1,d_1}$ such that $B_1 \in \Sigma_1$ (cf. (3.1)). Now, the singularities of B_1 are points $Q_1, \ldots, Q_l$ corresponding to the singularities $S_1, \ldots, S_l$ of B (via the natural birational correspondence between B_1 and B, which is biregular at those points) and the fundamental points M_0, M_1, M_2 of the Cremona transformation (each of the latter is an ordinary n-fold point of B_1).

Using the techniques of §3, we shall construct a subvariety S of $\mathbf{P}^{N_1}$ ($N_1 = \frac{1}{2}n_1(n_1 + 3)$, i.e., the space parametrizing curves of degree n_1), containing B_1 and having as generic member a curve with d nodes and an n-fold point at M_i, $i = 0, 1, 2$, such that the Cremona transforms (based at M_0, M_1, M_2) of the curves in S define the family (1.4) we are looking for. Next we give the details.

Let a general curve A_1 of Σ_1 (having exactly d_1 nodes) approach B_1 in such a way that each of the nodes of A_1 approaches a singularity of B_1. We will study how this can happen. (The meaning of these expressions is explained in (3.2), henceforth this terminology will be used without warning.) I claim: If suitably numbered, the nodes of A_1 can be split into four groups,

$$\{R^i_j\}, \quad i = 0, 1, 2,\ j = 1, \ldots, \tfrac{1}{2}n(n-1), \qquad \{R_j\}, \quad j = 1, \ldots, d,$$

where R^i_j approaches M_i, $i = 0, 1, 2$, and each R_j approaches a certain $Q_i \in B_1$ (i depends on j). To justify this, recall the meaning of the "δ invariant" associated to a singular point P of a curve C, $\delta = lg(\overline{\mathcal{O}_{C,P}}/\mathcal{O}_{C,P})$. It is the maximum number m such that there will be a family $\phi\colon Y \to T$ (T integral), where $\phi^{-1}(t_0)$ is an affine neighborhood $\mathcal{U}$ of P in C such that $\text{Sing}(\mathcal{U}) = \{P\}$ and $\phi^{-1}(t)$ has m distinct nodes, and no other singularity (cf. [**9**, I, §1; II, 5.2]). From this, if we divide the nodes of A_1 into four groups $\{R^i_j\}$, $i = 0, 1, 2$, $j = 1, \ldots, l_i$, $\{R_j\}$, $j = 1, \ldots, l_3$, as above, we shall get

$$l_i \leqslant \delta(M_i, B_1), \quad i = 0, 1, 2, \qquad l_3 \leqslant \sum_{j=1}^{l} \delta(Q_j, B_1).$$

But $\delta(M_i, B_i) = 3n(n-1)/2$ (M_i is an ordinary n-fold point) and $\Sigma_{j=1}^{l}\delta(Q_j, B_1) = \Sigma_{j=1}^{l}\delta(S_j, B) = d$ by the genus formula. Since $\Sigma_{i-0}^{3} l_i = 3n(n-1)/2 + d$ (the total number of nodes of A_1), each of the inequalities above is an equality, which proves our claim.

Now assign the points $R_1, \ldots, R_d$ on a suitable general $A_1 \in \Sigma_1$. By (3.1)(c), this determines an irreducible component Σ' of $V_{n_1,d}^{(0)}$, in particular $\Sigma_1 \subset \Sigma'$. Note that all these are subvarieties of $\mathbf{P}^{N_1}$, $N_1 = \frac{1}{2}(2n(2n+3))$, and $\dim \Sigma' = N_1 - d$. Now, inside $\mathbf{P}^{N_1}$ we have $L_i = \{D \subset P^2/D$ curve of degree n_1 having an n-fold point at $M_i\}$, $i = 0, 1, 2$. Each is a linear subvariety of $\mathbf{P}^{N_1}$, of dimension $\geqslant \frac{1}{2}n(n+1)$ (this is the number of conditions to get multiplicity n at M_i). Consider $(\bigcap_{i=0}^{2} L_i) \cap E'$. This contains B_1, and each irreducible component of it will have dimension

$$\rho \geqslant 2n(2n+3)/2 - d - 3n(n+1)/2 = (n^2+3n)/2 - d = 3n + p - 1.$$

Choose an irreducible component S containing B_1. The universal family on $V_{n_1,d}^{(0)}$ (cf. (3.1)(b)) induces a family of plane curves parametrized by S, all its members having multiplicity n at M_i, $i = 0, 1, 2$. Moreover, the general curve A' of E' specializes into the general curve D of S, which in its turn specializes to $B_1 \in S$. Since the d nodes of A' specialize to $Q_1, \ldots, Q_l$, D will have $m \geqslant l$ singularities $F_1, \ldots, F_m$, distinct from M_i, $i = 0, 1, 2$ (and $\delta' = \Sigma_{i=1}^{l}\delta_i(F_i, B) \geqslant d$), as well as points of multiplicity $\geqslant n$ at M_i, $i = 0, 1, 2$. Hence, $\delta(M_i, D) \geqslant n(n-1)/2$. Hence,

$$\begin{aligned} p_1 &= \text{genus of } D \leqslant (2n-1)(2n-2)/2 - d - 3n(n-1)/2 \\ &= (n-1)(n-2)/2 - d = p. \end{aligned}$$

Since B_1 has genus p, and by Theorem (1.2) we have $p \leqslant p_1$, we get $p_1 = p$.

Consider now the standard Cremona transformation of this family based at M_0, M_1, M_2. According to (3.4), we obtain a family of plane curves of degree 2, $2n - 3n = n$, of general genus p (the general curve being birationally equivalent to the previous one), the fiber at $B_1 \in S$ is B. The canonical morphism $\alpha: S \to \mathbf{P}^N$ is clearly injective, hence $T = \overline{\alpha(S)}$ (with the pull-back of the universal curve $\mathfrak{D} \subset \mathbf{P}^2 \times \mathbf{P}^N$) yields a family of plane curves, having B as a member, $T \subset \mathbf{P}^N$, general genus p and $\dim T \geqslant 3n + p - 1$. According to (3.3)(b) (Zariski's Theorem), the general curve of this family is a nodal curve. This proves (3.5).

(4.7) REMARK. A well-known open problem is: Using the notation of (3.1), is it true that there is a *single* irreducible component of $V_{n,\delta}^{(0)}$ whose general member is an irreducible curve? If the answer is "yes", then Theorem (1.4) would immediately imply: The closure, in $\mathbf{P}^N$, of $\{C: C$ is an irreducible plane curve of degree n and genus $p\}$ (p fixed) is an *irreducible* algebraic variety.

5. Proof of Theorem (1.3).

(5.1) Next we prove Theorem (1.2). In terms of "Severi's varieties" (cf. (3.1)), (1.3) is clearly equivalent to the following statement.

(5.2) Givn a reduced curve $B \subset \mathbf{P}^2$ with irreducible components $B_1, \ldots, B_r$ (we shall write in the sequel $B = B_1 + \cdots + B_r$), where $\deg B_i = n_i$, $g(B_i) = p_i$, $i = 1, \ldots, r$, and a number p, $p_1 + \cdots + p_r \leqslant p \leqslant (n-1)(n-2)/2$ (where $n = n_1 + \cdots + n_r$), then there is a number δ (necessarily equal to $(n-1)(n-2)/2 - p$)

and an irreducible component Σ of $V_{n,\delta}^{(0)}$ (cf. (3.1)(a)), Σ having as general member an irreducible curve, such that $B \in \Sigma$.

(5.3) We shall prove statement (5.2). We use the notation of (5.2).

By Theorem (1.3), if $\delta_i = (n_i - 1)(n_i - 2)/2 - p_i$, then there is an irreducible component Σ_i of $V_{n_i,\delta_i}^{(0)}$ such that

$$B_i \subset \Sigma_i. \tag{5.3.1}$$

Let C_i be a general member of Σ_i (it will be irreducible, because B_i is so). Consider $C = C_1 + \cdots + C_r$. By (3.3)(c), C will be a nodal curve. Let $Q_1, \ldots, Q_{\delta'}$ be all the nodes of C. The degree of C will be $\Sigma_{i=1}^r n_i = n$. It is easy to see that $B \in V_{n,\delta'}^{(0)}$ (it is easy to construct, using (5.3.1), a family with integral parameter space, showing that C specializes to B, then use the universal property of (3.1)(b). Moreover, C is in a single irreducible component of $V_{n,\delta'}^{(0)}$, say Σ' ($V_{n,\delta'}^{(0)}$ is smooth at C).

Now take any integer h, $0 \leqslant h \leqslant (n-1)(n-2)/2 - \Sigma_{i=1}^r p_i$. Clearly, we may choose nodes of C, say (after reordering them, if necessary) $Q_1, \ldots, Q_{r-1+h}$ forming a "connection set", i.e., $C - \{Q_{r+h}, \ldots, Q_{\delta'}\}$ is connected. Consider the assignment $\{Q_{r+h}, \ldots, Q_{\delta'}\}$ on C (cf. (3.1)(b)). According to (3.1)(b), there is an irreducible component Σ of $V_{n,\delta}^{(0)}$, $\delta = \delta' - r - h + 1$, whose general member is irreducible, such that Σ contains Σ'. In particular, $B \in \Sigma$. We claim: The genus p of the general curve of Σ is $p_1 + \cdots + p_r + h$. This is a consequence of an elementary calculation, using the fact that $p = (n-1)(n-2)/2 - \delta$, $\delta = \delta' - r - h + 1$, $\delta' = \Sigma_{i=1}^r \delta_i + \Sigma_{i<j} n_i n_j$, $n = \Sigma_{i=1}^r n_i$, $p_i = (n_i - 1)(n_i - 2)/2 - \delta_i$.

Hence, (5.2) has been proved.

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